Computers and Mathematics with Applications 62 (2011) 1442–1450

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Controllability of fractional evolution nonlocal impulsive quasilinear delay integro-differential systems

Amar Debbouche [a], Dumitru Baleanu [b,c,*]

[a] Department of Mathematics, Faculty of Science, Guelma University, Guelma, Algeria
[b] Department of Mathematics and Computer Science, Faculty of Arts and Science, Cankaya University, Ankara, Turkey
[c] Institute of Space Sciences, P.O. Box, MG-23, R 76900, Magurele-Bucharest, Romania

ARTICLE INFO

Keywords:
Fractional integro-differential systems
Controllability
(α, u)-resolvent family
Nonlocal and impulsive conditions
Fixed point theorem

ABSTRACT

In this work, the controllability result of a class of fractional evolution nonlocal impulsive quasilinear delay integro-differential systems in a Banach space has been established by using the theory of fractional calculus, fixed point technique and also we introduced a new concept called (α, u)-resolvent family. As an application that illustrates the abstract results, an example is given.

1. Introduction

The purpose of this paper is to study the fractional nonlocal impulsive integro-differential control system of the form

$$\frac{\mathrm{d}^\alpha u(t)}{\mathrm{d}t^\alpha} + A(t, u(t))u(t) = (B\mu)(t) + \Phi\left(t, f(t, u(\beta(t))), \int_0^t g(t, s, u(\gamma(s)))\mathrm{d}s\right), \tag{1.1}$$

$$u(0) + h(u) = u_0, \tag{1.2}$$

$$\Delta u(t_i) = I_i(u(t_i)), \tag{1.3}$$

where the state $u(.)$ takes values in the Banach space X, $0 < \alpha \le 1$, $t \in [0, a]$, $u_0 \in X$, $i = 1, 2, \ldots, m$ and $0 < t_1 < t_2 < \cdots < t_m < a$. We assume that $-A(t, .)$ is a closed linear operator defined on a dense domain $D(A)$ in X into X such that $D(A)$ is independent of t. It is assumed also that $-A(t, .)$ generates an evolution operator in the Banach space X, the control function μ belongs to the spaces $L^2(J, U)$, a Banach space of admissible control functions with U as a Banach space and $B : U \to X$ is a bounded linear operator. The functions $f : J \times X^r \to X$, $g : \Lambda \times X^k \to X$, $\Phi : J \times X^2 \to X$, $h : PC(J, X) \to X$, $u(\beta) = (u(\beta_1), \ldots, u(\beta_r))$, $u(\gamma) = (u(\gamma_1), \ldots, u(\gamma_k))$, and $\beta_p, \gamma_q : J \to J$ are given, where $p = 1, 2, \ldots, r$, $q = 1, 2, \ldots, k$. Here $J = [0, a]$ and $\Lambda = \{(t, s) : 0 \le s \le t \le a\}$.

Let $PC(J, X)$ consist of functions u from J into X, such that $u(t)$ is continuous at $t \ne t_i$ and left continuous at $t = t_i$ and the right limit $u(t_i^+)$ exists for $i = 1, 2, \ldots, m$. Clearly $PC(J, X)$ is a Banach space with the norm $\|u\|_{PC} = \sup_{t\in J} \|u(t)\|$, and let $\Delta u(t_i) = u(t_i^+) - u(t_i^-)$ constitute an impulsive condition.

In recent years, fractional differential equations have attracted the attention of many mathematician and physicists; see for instance, Baleanu et al. [1–3], Agarwal and Lakshmikantham [4], basic papers of Lakshmikantham et al. [5–8] and Kilbas et al. [9,10]; see also [11–16]. The existence results to evolution equations with nonlocal conditions in Banach space was studied first by Byszewski [17,18]. Deng [19] indicated that, using the nonlocal condition $u(0) + h(u) = u_0$ to describe for

* Corresponding author at: Department of Mathematics and Computer Science, Faculty of Arts and Science, Cankaya University, Ankara, Turkey.
E-mail addresses: amar_debbouche@yahoo.fr (A. Debbouche), dumitru@cankaya.edu.tr (D. Baleanu).

0898-1221/$ – see front matter
doi:10.1016/j.camwa.2011.03.075

instance, the diffusion phenomenon of a small amount of gas in a transparent tube can give better result than using the usual local Cauchy problem $u(0) = u_0$. Let us observe also that since Deng's papers, the function h is considered

$$h(u) = \sum_{k=1}^{p} c_k u(t_k), \tag{1.4}$$

where c_k, $k = 1, 2, \dots, p$ are given constants and $0 \le t_1 < \cdots < t_p \le a$.

Differential equations involving impulsive effects occur in many applications: pharmacokinetics, the radiation of electromagnetic waves, population dynamics, biological systems, the abrupt increase of glycerol in fed-batch culture, bio-technology, nanoelectronics, etc., [20]. For the basic theory of impulsive differential equations the reader can refer to [21].

Controllability problems for different kinds of dynamical systems have been considered in many papers [20,22–26]. There are few papers discussing controllability results for impulsive functional differential inclusions in Banach spaces; see [27,28]. Sakthivel et al. [29] investigated approximate controllability of impulsive differential equations with state-dependent delay. Jeong et al. [30] studied controllability for semilinear retarded control systems in Hilbert spaces. Tai and Wang [20] gave sufficient conditions for the controllability of fractional impulsive neutral functional integro-differential systems in a Banach space. Balachandran and Park [23] studied the controllability of fractional integro-differential systems in Banach spaces.

In the past decade, many authors investigated the existence result for fractional evolution equations; see [31,32]. Moreover, there are different types of mild solutions that have been proved. For example, the first one was constructed in terms of a probability density function given by El-Borai [33] and was then developed by Zhou et al. [34,35], and the second one was presented in terms of an α-resolvent family provided by Araya et al. [36] and then Mophou et al. [37]. But, in both senses, if the closed operator in the evolution equation is dependent on more than variable, then the considered case can be taken as an open problem. For this reason, we will introduce in this article a new concept called (α, u)-resolvent family, which is based on Araya–Lizama concepts [36], and Hille–Phillips principles [38]. Our paper is organized as follows. Section 2 is devoted to a review of some essential results in fractional calculus and the resolvent operators that will be used in this work to obtain our main results. In Section 3, we state and prove the controllability result. Section 4 deals with an example to illustrate the abstract results.

2. Preliminaries

Let X and Y be two Banach spaces such that Y is densely and continuously embedded in X. For any Banach space Z, the norm of Z is denoted by $\|.\|_Z$. The space of all bounded linear operators from X to Y is denoted by $B(X, Y)$ and $B(X, X)$ is written as $B(X)$. We recall some definitions in fractional calculus from [39,40], then some known facts of the theory of semigroups from [41,42].

Definition 2.1. The fractional integral of order $\alpha > 0$ is defined by

$$I_a^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) \mathrm{d}s$$

where Γ is the gamma function and $f \in L^1([a, b], \mathbb{R}^+)$.

If $a = 0$, we can write $I^{\alpha} f(t) = (g_\alpha * f)(t)$, where

$$g_\alpha(t) := \begin{cases} \dfrac{1}{\Gamma(\alpha)} t^{\alpha-1}, & t > 0, \\ 0, & t \le 0, \end{cases}$$

as usual, $*$ denotes the convolution of functions, also we have $\lim_{\alpha \to 0} g_\alpha(t) = \delta(t)$, which is the delta function.

The Riemann–Liouville fractional derivative of order $n - 1 < \alpha < n$ is defined by

$${}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_a^t (t-s)^{n-\alpha-1} f(s) \mathrm{d}s,$$

where f is an abstract continuous function on the interval $[a, b]$ and $n \in \mathbb{N}^*$, also the Caputo fractional derivative of order $n - 1 < \alpha < n$ is defined by

$${}_a^cD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s) \mathrm{d}s.$$

This definition is still till now a basic source for all authors that are working in the field of fractional calculus.

Definition 2.2. A two parameter family of bounded linear operators $Q(t, s)$, $0 \le s \le t \le a$, on X is called an evolution system if the following two conditions are satisfied

(i) $Q(t, t) = I$, $Q(t, r)Q(r, s) = Q(t, s)$ for $0 \le s \le r \le t \le a$,
(ii) $(t, s) \to Q(t, s)$ is strongly continuous for $0 \le s \le t \le a$.

More details about evolution system and quasilinear equation of evolution can be found in [41, Chapter 5 and Section 6.4 respectively].

Let E be the Banach space formed from $D(A)$ with the graph norm. Since $-A(t)$ is a closed operator, it follows that $-A(t)$ is in the set of bounded operators from E to X.

Definition 2.3. Let $A(t, u)$ be a closed and linear operator with domain $D(A)$ defined on a Banach space X and $\alpha > 0$. Let $\rho[A(t, u)]$ be the resolvent set of $A(t, u)$. We call $A(t, u)$ the generator of an (α, u)-resolvent family if there exist $\omega \geq 0$ and a strongly continuous function $R_{(\alpha,u)} : \mathbb{R}^2_+ \to L(X)$ such that $\{\lambda^\alpha : \mathrm{Re}(\lambda) > \omega\} \subset \rho(A)$ and for $0 \leq s \leq t < \infty$,

$$(\lambda^\alpha I - A(s, u))^{-1} v = \int_0^\infty e^{-\lambda(t-s)} R_{(\alpha,u)}(t, s) v \, \mathrm{d}t, \quad \mathrm{Re}(\lambda) > \omega, (u, v) \in X^2. \tag{2.1}$$

In this case, $R_{(\alpha,u)}(t, s)$ is called the (α, u)-resolvent family generated by $A(t, u)$.

Remark 2.1. 1. In the deleting case of s and u, (2.1) will be reduced to the introduced concept by [36].
2. We can deduce that (1.1)–(1.3) is well posed if and only if, $-A(t, u)$ is the generator of (α, u)-resolvent family.
3. Here, $R_{(\alpha,u)}(t, s)$ can be extracted from the evolution operator of the generator $-A(t, u)$.
4. The (α, u)-resolvent family is similar to the evolution operator for nonautonomous differential equations in a Banach space.

Let Ω be a subset of X.

Definition 2.4 (*Compare [11,20,22] with [36]*)**.** By a mild solution of (1.1)–(1.3) we mean a function $u \in PC(J : X)$ with values in Ω satisfying the integral equation

$$\begin{aligned} u_\mu(t) &= R_{(\alpha,u)}(t, 0)u_0 - R_{(\alpha,u)}(t, 0)h(u) \\ &\quad + \int_0^t R_{(\alpha,u)}(t, s) \left[(B\mu)(s) + \Phi\left(s, f(s, u(\beta(s))), \int_0^s g(s, \eta, u(\gamma(\eta)))\mathrm{d}\eta\right)\right] \mathrm{d}s \\ &\quad + \sum_{0<t_i<t} R_{(\alpha,u)}(t, t_i) I_i(u(t_i)), \quad t \in J \end{aligned} \tag{2.2}$$

for all $u_0 \in X$ and admissible control $\mu \in L^2(J, U)$. We assume the following conditions.
(H_1) The bounded linear operator $W : L^2(J, U) \to X$ defined by

$$W\mu = \int_0^a R_{(\alpha,u)}(a, s) B\mu(s) \mathrm{d}s,$$

has an induced inverse operator $\tilde{W}^{-1}$ which takes values in $L^2(J, U)/\ker W$ and there exist positive constants M_1, M_2, such that $\|B\| \leq M_1$ and $\|\tilde{W}^{-1}\| \leq M_2$.
(H_2) $h : PC(J : \Omega) \to Y$ is Lipschitz continuous in X and bounded in Y, that is, there exist constants $k_1 > 0$ and $k_2 > 0$ such that

$$\begin{aligned} &\|h(u)\|_Y \leq k_1, \\ &\|h(u) - h(v)\|_Y \leq k_2 \max_{t \in J} \|u - v\|_{PC}, \quad u, v \in PC(J : X). \end{aligned}$$

For conditions (H_3)–(H_5) let Z be taken as both X and Y.
(H_3) $g : \Lambda \times Z^k \to Z$ is continuous and there exist constants $k_3 > 0$ and $k_4 > 0$ such that

$$\begin{aligned} &\int_0^t \|g(t, s, u_1, \ldots, u_k) - g(t, s, v_1, \ldots, v_k)\|_Z \mathrm{d}s \leq k_3 \sum_{q=1}^k \|u_q - v_q\|_Z, \quad u_q, v_q \in X, q = 1, 2, \ldots, k, \\ &k_4 = \max\left\{\int_0^t \|g(t, s, 0, \ldots, 0)\|_Z \mathrm{d}s : \quad (t, s) \in \Lambda\right\}. \end{aligned}$$

(H_4) $f : J \times Z^r \to Z$ is continuous and there exist constants $k_5 > 0$ and $k_6 > 0$ such that

$$\begin{aligned} &\|f(t, u_1, \ldots, u_r) - f(t, v_1, \ldots, v_r)\|_Z \leq k_5 \sum_{p=1}^r \|u_p - v_p\|_Z, \quad u_p, v_p \in X, p = 1, 2, \ldots, r, \\ &k_6 = \max_{t \in J} \|f(t, 0, \ldots, 0)\|_Z. \end{aligned}$$

(H_5) $\Phi : J \times Z^2 \to Z$ is continuous and there exist constants $k_7 > 0$ and $k_8 > 0$ such that

$$\|\Phi(t, u_1, u_2) - \Phi(t, v_1, v_2)\|_Z \le k_7(\|u_1 - v_1\|_Z + \|u_2 - v_2\|_Z), \quad u_1, u_2, v_1, v_2 \in X,$$
$$k_8 = \max_{t\in J} \|\Phi(t, 0, 0)\|_Z.$$

(H_6) $\beta_p, \gamma_q : J \to J$ are bijective absolutely continuous and there exist constants $c_p > 0$ and $b_q > 0$ such that $\beta_p'(t) \ge c_p$ and $\gamma_q'(t) \ge b_q$ respectively for $t \in J$, $p = 1, \dots, r$ and $q = 1, \dots, k$.

(H_7) $I_i : X \to X$ are continuous and there exist constants $l_i > 0$, $i = 1, 2, \dots, m$ such that

$$\|I_i(u) - I_i(v)\| \le l_i\|u - v\|, \quad u, v \in X.$$

Let us take $M_0 = \max \|R_{(\alpha,u)}(t, s)\|_{B(Z)},\ 0 \le s \le t \le a, u \in \Omega$.

(H_8) There exist positive constants $\delta_1, \delta_2, \delta_3 \in (0, \delta/3]$ and $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in [0, \frac{1}{4})$ such that

$$\delta_1 = M_0\|u_0\| + M_0k_1,$$
$$\delta_2 = M_0M_1M_2[\|u_1\| + M_0\|u_0\| + M_0k_1 + M_0k_7\theta + M_0k_8a + M_0\xi]a,$$
$$\delta_3 = M_0k_7\theta + M_0k_8a + M_0\xi,$$

and

$$\lambda_1 = Ka\|u_0\| + k_1Ka + M_0k_2,$$
$$\lambda_2 = 2a^2KM_1M_2\{\|u_1\|_Y + M_0(\|u_0\|_Y + k_1 + k_7\theta + k_8a + \xi)\},$$
$$\lambda_3 = Ka(k_7\theta + k_8a) + M_0k_7\rho,$$
$$\lambda_4 = Ka\xi + M_0\sum_{i=1}^{m} l_i,$$

where $\rho = a[k_5(1/c_1 + \cdots + 1/c_r) + k_3(1/b_1 + \cdots + 1/b_k)]$, $\theta = \rho\delta + a(k_4 + k_6)$ and $\xi = \sum_{i=1}^{m}(l_i\delta + \|I_i(0)\|)$.

3. Controllability result

Definition 3.1. We shall say that the fractional system (1.1)–(1.3) is controllable on the interval J if for all $u_0, u_1 \in X$, there exists a control $\mu \in L^2(J, U)$, such that the mild solution $u(.)$ of (1.1)–(1.3) corresponding to μ, verifies: $u(0) + h(u) = u_0$, $\Delta u(t_i) = I_i(u(t_i))$, $i = 1, 2, \dots, m$ and $u_\mu(a) = u_1$.

Lemma 3.1. *Let $R_{(\alpha,u)}(t, s)$ be the (α, u)-resolvent family for the fractional problem (1.1)–(1.3). There exists a constant $K > 0$ such that*

$$\|R_{(\alpha,u)}(t, s)\omega - R_{(\alpha,v)}(t, s)\omega\| \le K\|\omega\|_Y \int_s^t \|u(\tau) - v(\tau)\|\mathrm{d}\tau,$$

for every $u, v \in PC(J : X)$ with values in Ω and every $\omega \in Y$.

Proof. Since the resolvent operator is similarly to the evolution operator for nonautonomous differential equations in a Banach space, then we can use a similar manner as in [41, Lemma 4.4, p. 202]. □

Let $S_\delta = \{u : u \in PC(J : X), u(0) + h(u) = u_0, \Delta u(t_i) = I_i(u(t_i)),\ \|u\| \le \delta\}$, for $t \in J$, $\delta > 0$, $u_0 \in X$ and $i = 1, \dots, m$.

Lemma 3.2.

$$\|\varphi(t)\| \le \theta,$$

where

$$\varphi(t) = \int_0^t \left[f(s, u(\beta(s))) + \int_0^s g(s, \tau, u(\gamma(\tau)))\mathrm{d}\tau\right]\mathrm{d}s.$$

Proof. We have

$$\|\varphi(t)\| \le \int_0^t \Bigg[\|f(s, u(\beta_1(s)), \dots, u(\beta_r(s))) - f(s, 0, \dots, 0)\| + \|f(s, 0, \dots, 0)\|$$
$$+ \int_0^s \|g(s, \tau, u(\gamma_1(\tau)), \dots, u(\gamma_k(\tau))) - g(s, \tau, 0, \dots, 0)\|\mathrm{d}\tau + \int_0^s \|g(s, \tau, 0, \dots, 0)\|\mathrm{d}\tau\Bigg]\mathrm{d}s.$$

Using H_3, H_4 and H_6, we get

$$\begin{aligned}
\|\varphi(t)\| &\leq \int_0^t [k_5(\|u(\beta_1(s))\| + \cdots + \|u(\beta_r(s))\|) + k_6 + k_3(\|u(\gamma_1(s))\| + \cdots + \|u(\gamma_k(s))\|) + k_4]\mathrm{d}s \\
&\leq \int_0^t [k_5\{\|u(\beta_1(s))\|(\beta_1'(s)/c_1) + \cdots + \|u(\beta_r(s))\|(\beta_r'(s)/c_r)\} + k_6 \\
&\quad + k_3\{\|u(\gamma_1(s))\|(\gamma_1'(s)/b_1) + \cdots + \|u(\gamma_k(s))\|(\gamma_k'(s)/b_k)\} + k_4]\mathrm{d}s \\
&\leq \frac{k_5}{c_1}\int_{\beta_1(0)}^{\beta_1(t)} \|u(\tau)\|\mathrm{d}\tau + \cdots + \frac{k_5}{c_r}\int_{\beta_r(0)}^{\beta_r(t)} \|u(\tau)\|\mathrm{d}\tau + k_6 a \\
&\quad + \frac{k_3}{b_1}\int_{\gamma_1(0)}^{\gamma_1(t)} \|u(\eta)\|\mathrm{d}\eta + \cdots + \frac{k_3}{b_k}\int_{\gamma_k(0)}^{\gamma_k(t)} \|u(\eta)\|\mathrm{d}\eta + k_4 a.
\end{aligned}$$

Hence the required result. □

Theorem 3.3. *Suppose that the operator $-A(t,u)$ generates an (α,u)-resolvent family with $\|R_{(\alpha,u)}(t,s)\| \leq Me^{-\sigma(t-s)}$ for some constants $M,\sigma > 0$. If hypotheses (H_1)–(H_8) are satisfied, then the fractional control integro-differential system (1.1) with nonlocal condition (1.2) and impulsive condition (1.3) is controllable on J.*

Proof. Using hypothesis (H_1), for an arbitrary function $u(.)$, we define the control

$$\begin{aligned}
\mu(t) &= \tilde{W}^{-1}\Bigg[u_1 - R_{(\alpha,u)}(a,0)u_0 + R_{(\alpha,u)}(a,0)h(u) \\
&\quad - \int_0^a R_{(\alpha,u)}(a,s)\Phi\left(s, f(s,u(\beta(s))), \int_0^s g(s,\eta,u(\gamma(\eta)))\mathrm{d}\eta\right)\mathrm{d}s - \sum_{i=1}^m R_{(\alpha,u)}(a,t_i)I_i(u(t_i))\Bigg](t).
\end{aligned}$$

We define an operator $P : S_\delta \to S_\delta$ by

$$\begin{aligned}
(Pu_\mu)(t) &= R_{(\alpha,u)}(t,0)u_0 - R_{(\alpha,u)}(t,0)h(u) + \int_0^t R_{(\alpha,u)}(t,\eta)B\tilde{W}^{-1}\Bigg[u_1 - R_{(\alpha,u)}(a,0)u_0 + R_{(\alpha,u)}(a,0)h(u) \\
&\quad - \int_0^a R_{(\alpha,u)}(a,s)\Phi\left(s, f(s,u(\beta(s))), \int_0^s g(s,\tau,u(\gamma(\tau)))\mathrm{d}\tau\right)\mathrm{d}s - \sum_{i=1}^m R_{(\alpha,u)}(a,t_i)I_i(u(t_i))\Bigg](\eta)\mathrm{d}\eta \\
&\quad + \int_0^t R_{(\alpha,u)}(t,s)\Phi\left(s, f(s,u(\beta(s))), \int_0^s g(s,\tau,u(\gamma(\tau)))\mathrm{d}\tau\right)\mathrm{d}s + \sum_{0<t_i<t} R_{(\alpha,u)}(t,t_i)I_i(u(t_i)).
\end{aligned}$$

Using this controller we shall show that the operator P has a fixed point. This fixed point is then a solution of Eq. (2.1).

Clearly $Pu_\mu(a) = u_1$, which means that the control μ steers system (1.1)–(1.3) from the initial state u_0 to u_1 in time a, provided we can obtain a fixed point of the nonlinear operator P.

Now we show that P maps S_δ into itself.

$$\begin{aligned}
\|(Pu_\mu)(t)\| &\leq \|R_{(\alpha,u)}(t,0)u_0\| + \|R_{(\alpha,u)}(t,0)h(u)\| \\
&\quad + \int_0^t \|R_{(\alpha,u)}(t,\eta)\|\,\|B\tilde{W}^{-1}\|\Bigg[\|u_1\| + \|R_{(\alpha,u)}(a,0)u_0\| + \|R_{(\alpha,u)}(a,0)h(u)\| + \int_0^a \|R_{(\alpha,u)}(a,s)\| \\
&\quad \times \left\{\left\|\Phi\left(s, f(s,u(\beta(s))), \int_0^s g(s,\tau,u(\gamma(\tau)))\mathrm{d}\tau\right) - \Phi(s,0,0)\right\| + \|\Phi(s,0,0)\|\right\}\mathrm{d}s \\
&\quad + \sum_{i=1}^m \|R_{(\alpha,u)}(a,t_i)\|\{\|I_i(u(t_i)) - I_i(0)\| + \|I_i(0)\|\}\Bigg]\mathrm{d}\eta + \int_0^t \|R_{(\alpha,u)}(t,s)\| \\
&\quad \times \left\{\left\|\Phi(s, f(s,u(\beta(s))), \int_0^s g(s,\tau,u(\gamma(\tau)))\mathrm{d}\tau) - \Phi(s,0,0)\right\| + \|\Phi(s,0,0)\|\right\}\mathrm{d}s \\
&\quad + \sum_{0<t_i<t} \|R_{(\alpha,u)}(t,t_i)\|\{\|I_i(u(t_i)) - I_i(0)\| + \|I_i(0)\|\}.
\end{aligned}$$

Using H_1, H_2, H_5 and H_7, we get

$$\begin{aligned}\|(Pu_\mu)(t)\| \le{}& M_0\|u_0\| + M_0k_1 + \int_0^t M_0M_1M_2\Bigg[\|u_1\| + M_0\|u_0\| + M_0k_1 \\ &+ \int_0^a M_0\left\{k_7\left(\|f(s,u(\beta(s)))\| + \left\|\int_0^s g(s,\tau,u(\gamma(\tau)))\mathrm{d}\tau\right\|\right) + k_8\right\}\mathrm{d}s + M_0\sum_{i=1}^m (l_i\delta + \|I_i(0)\|)\Bigg]\mathrm{d}\eta \\ &+ \int_0^t M_0\left\{k_7\left(\|f(s,u(\beta(s)))\| + \left\|\int_0^s g(s,\tau,u(\gamma(\tau)))\mathrm{d}\tau\right\|\right) + k_8\right\}\mathrm{d}s + M_0\sum_{i=1}^m (l_i\delta + \|I_i(0)\|).\end{aligned}$$

According to Lemma 3.2 and H_8, we have

$$\begin{aligned}\|(Pu_\mu)(t)\| \le{}& M_0\|u_0\| + M_0k_1 + M_0M_1M_2[\|u_1\| + M_0\|u_0\| + M_0k_1 + M_0k_7\theta + M_0k_8a + M_0\xi]a \\ &+ M_0k_7\theta + M_0k_8a + M_0\xi.\end{aligned}$$

From assumption H_8, one gets $\|(Pu_\mu)(t)\| \le \delta$. Thus, P maps S_δ into itself.

Now for $u, v \in S_\delta$, we have

$$\|Pu_\mu(t) - Pv_\mu(t)\| \le I_1 + I_2 + I_3 + I_4,$$

where

$$I_1 = \|R_{(\alpha,u)}(t,0)u_0 - R_{(\alpha,v)}(t,0)u_0\| + \|R_{(\alpha,u)}(t,0)h(u) - R_{(\alpha,v)}(t,0)h(v)\|,$$

$$\begin{aligned}I_2 ={}& \int_0^t \Bigg\{\Bigg\|R_{(\alpha,u)}(t,\eta)B\tilde{W}^{-1}\Bigg[u_1 - R_{(\alpha,u)}(a,0)u_0 + R_{(\alpha,u)}(a,0)h(u) \\ &- \int_0^a R_{(\alpha,u)}(a,s)\Phi\left(s, f(s,u(\beta(s))), \int_0^s g(s,\tau,u(\gamma(\tau)))\mathrm{d}\tau\right)\mathrm{d}s - \sum_{i=1}^m R_{(\alpha,u)}(a,t_i)I_i(u(t_i))\Bigg] \\ &- R_{(\alpha,v)}(t,\eta)B\tilde{W}^{-1}\Bigg[u_1 - R_{(\alpha,v)}(a,0)u_0 + R_{(\alpha,v)}(a,0)h(v) \\ &- \int_0^a R_{(\alpha,v)}(a,s)\Phi\left(s, f(s,v(\beta(s))), \int_0^s g(s,\tau,v(\gamma(\tau)))\mathrm{d}\tau\right)\mathrm{d}s - \sum_{i=1}^m R_{(\alpha,v)}(a,t_i)I_i(v(t_i))\Bigg]\Bigg\|\Bigg\}\mathrm{d}\eta,\end{aligned}$$

$$\begin{aligned}I_3 ={}& \int_0^t \Bigg\|R_{(\alpha,u)}(t,s)\Phi\left(s, f(s,u(\beta(s))), \int_0^s g(s,\tau,u(\gamma(\tau)))\mathrm{d}\tau\right) \\ &- R_{(\alpha,v)}(t,s)\Phi\left(s, f(s,v(\beta(s))), \int_0^s g(s,\tau,v(\gamma(\tau)))\mathrm{d}\tau\right)\Bigg\|\mathrm{d}s,\end{aligned}$$

and

$$I_4 = \sum_{i=1}^m \|R_{(\alpha,u)}(t,t_i)I_i(u(t_i)) - R_{(\alpha,v)}(t,t_i)I_i(v(t_i))\|.$$

Applying Lemma 3.1 and H_2, we get

$$\begin{aligned}I_1 \le{}& \|R_{(\alpha,u)}(t,0)u_0 - R_{(\alpha,v)}(t,0)u_0\| + \|R_{(\alpha,u)}(t,0)h(u) - R_{(\alpha,v)}(t,0)h(u)\| + \|R_{(\alpha,v)}(t,0)h(u) - R_{(\alpha,v)}(t,0)h(v)\| \\ \le{}& \{Ka\|u_0\| + k_1Ka + M_0k_2\}\max_{\tau\in J}\|u(\tau) - v(\tau)\|.\end{aligned}$$

Also, we apply Lemmas 3.1 and 3.2, H_1, H_2, H_5 and H_8, we obtain

$$\begin{aligned}I_2 \le{}& a^2KM_1M_2\Bigg\{\Bigg\|2\max\Bigg(\Bigg[u_1 - R_{(\alpha,u)}(a,0)u_0 + R_{(\alpha,u)}(a,0)h(u) \\ &- \int_0^a R_{(\alpha,u)}(a,s)\Phi\left(s, f(s,u(\beta(s))), \int_0^s g(s,\tau,u(\gamma(\tau)))\mathrm{d}\tau\right)\mathrm{d}s - \sum_{i=1}^m R_{(\alpha,u)}(a,t_i)\{I_i(u(t_i)) - I_i(0) + I_i(0)\}\Bigg],\end{aligned}$$

$$
\left[u_1 - R_{(\alpha,v)}(a,0)u_0 + R_{(\alpha,v)}(a,0)h(v) - \int_0^a R_{(\alpha,v)}(a,s)\Phi\left(s, f(s,v(\beta(s))), \int_0^s g(s,\tau,v(\gamma(\tau)))\mathrm{d}\tau\right)\mathrm{d}s\right.
$$

$$
\left.\left.\left.- \sum_{i=1}^m R_{(\alpha,v)}(a,t_i)\{I_i(v(t_i)) - I_i(0) + I_i(0)\}\right]\right)\right\|_Y \Bigg\} \max_{\tau\in J}\|u(\tau)-v(\tau)\|
$$

$$
\le 2a^2 K M_1 M_2\{\|u_1\|_Y + M_0(\|u_0\|_Y + k_1 + k_7\theta + k_8 a + \xi)\} \max_{\tau\in J}\|u(\tau)-v(\tau)\|.
$$

Again, Lemmas 3.1 and 3.2, H_3–H_6 and H_8 imply that

$$
I_3 \le \int_0^t \left\{\left\|R_{(\alpha,u)}(t,s)\Phi\left(s, f(s,u(\beta(s))), \int_0^s g(s,\tau,u(\gamma(\tau)))\mathrm{d}\tau\right)\right.\right.
$$

$$
\left. - R_{(\alpha,v)}(t,s)\Phi\left(s, f(s,u(\beta(s))), \int_0^s g(s,\tau,u(\gamma(\tau)))\mathrm{d}\tau\right)\right\|
$$

$$
+ \left\|R_{(\alpha,v)}(t,s)\Phi\left(s, f(s,u(\beta(s))), \int_0^s g(s,\tau,u(\gamma(\tau)))\mathrm{d}\tau\right)\right.
$$

$$
\left.\left. - R_{(\alpha,v)}(t,s)\Phi\left(s, f(s,v(\beta(s))), \int_0^s g(s,\tau,v(\gamma(\tau)))\mathrm{d}\tau\right)\right\|\right\}\mathrm{d}s
$$

$$
\le Ka(k_7\theta + k_8 a)\max_{\tau\in J}\|u(\tau)-v(\tau)\| + M_0 k_7 \int_0^t \Bigg\{\|f(s,u(\beta(s)) - f(s,v(\beta(s))))\|
$$

$$
+ \int_0^s \|g(s,\tau,u(\gamma(\tau))) - g(s,\tau,v(\gamma(\tau)))\|\mathrm{d}\tau\Bigg\}\mathrm{d}s
$$

$$
\le Ka(k_7\theta + k_8 a)\max_{\tau\in J}\|u(\tau)-v(\tau)\| + M_0 k_7 \int_0^t \Bigg\{k_5 \sum_{p=1}^r \|u(\beta_p(s)) - v(\beta_p(s))\|(\beta_p'(s)/c_p)
$$

$$
+ k_3 \sum_{q=1}^k \|u(\gamma_q(s)) - v(\gamma_q(s))\|(\gamma_q'(s)/b_q)\Bigg\}\mathrm{d}s
$$

$$
\le \{Ka(k_7\theta + k_8 a) + M_0 k_7 \rho\}\max_{\tau\in J}\|u(\tau)-v(\tau)\|.
$$

Now, from Lemma 3.1, H_7 and H_8, we have

$$
I_4 \le \sum_{i=1}^m \{\|R_{(\alpha,u)}(t,t_i)I_i(u(t_i)) - R_{(\alpha,v)}(t,t_i)I_i(u(t_i))\| + \|R_{(\alpha,v)}(t,t_i)I_i(u(t_i)) - R_{(\alpha,v)}(t,t_i)I_i(v(t_i))\|\}
$$

$$
\le \left\{K\sum_{i=1}^m (l_i\delta + \|I_i(0)\|)a + M_0\sum_{i=1}^m l_i\right\}\max_{\tau\in J}\|u(\tau)-v(\tau)\|.
$$

It follows from these estimations that

$$
\|Pu_\mu(t) - Pv_\mu(t)\| \le \sum_{j=1}^4 I_j \le \sum_{j=1}^4 \lambda_j \max_{\tau\in J}\|u(\tau)-v(\tau)\|.
$$

Therefore, P is a contraction mapping and hence there exists a unique fixed point $u\in X$, such that $Pu(t)=u(t)$. Any fixed point of P is a mild solution of (1.1)–(1.3) on J which satisfies $u(a)=u_1$. Thus, system (1.1)–(1.3) is controllable on J. □

Remark 3.1. Let $u_a(u_0 - h(u); \mu)$ be the state value of (1.1)–(1.3) at terminal time a corresponding to the control μ and the nonlocal initial value $u_0 - h(u) \in P$ is an abstract phase space described axiomatically. Introduce the set

$$
\Re(a, u_0 - h(u)) = \{u_a(u_0 - h(u); \mu)(0) : \mu \in L_2(J,U)\},
$$

which is called the reachable set of system (1.1)–(1.3) at terminal time a and its closure in X is denoted by $\overline{\Re(a, u_0 - h(u))}$. The fractional nonlocal impulsive control system (1.1)-(1.3) is said to be approximately controllable on the interval J if $\overline{\Re(a, u_0 - h(u))} = X$; see [22,29].

4. Example

Consider the fractional nonlocal impulsive integro-partial differential control system of the form

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + a(x,t,u(x,t))\frac{\partial^2 u(x,t)}{\partial x^2} = \zeta(x,t) + \Phi\left(t, x \arctan \varphi_p(x,t,u), \int_0^t e^{-\varphi_q(x,s,u)} \mathrm{d}s\right), \tag{4.1}$$

$$u(x,0) + \sum_{k=1}^{m} c_k u(x,t_k) = u_0(x), \quad x \in [0,\pi], \tag{4.2}$$

$$u(0,t) = u(\pi,t) = 0, \quad t \in J, \tag{4.3}$$

$$\Delta u(t_k, x) = -u(t_k, x), \quad x \in (0,1), k = 1, \ldots, m, \tag{4.4}$$

where $0 < \alpha \le 1, 0 < t_1 < \cdots < t_m < a, \Phi, p, q$ as above and the function $a(x, t, .)$ is continuous. Let us take

$$X = L^2[0,\pi], \qquad PC = PC(J, S_\delta), \qquad S_\delta = \{y \in L^2[0,\pi] : \|y\| \le \delta\}.$$

Put $(B\mu)(x,t) = \zeta(x,t),\ x \in [0,\pi]$ where $\mu(t) = \zeta(.,t)$ and $\zeta : [0,\pi] \times J \to [0,\pi]$ is continuous, $h(u(.,t)) = \sum_{k=1}^{m} c_k u(., t_k)$, the functions $f(t, u(\beta(t))) = x \arctan \varphi_p(x,t,u)$ and $g(t,s,u(\gamma(t))) = e^{-\varphi_q(x,s,u)}$, where $\varphi_\eta(x,t,u) = (u(x, \sin t), u(x, (\sin t)/2), \ldots, u(x, (\sin t)/\eta))$, and $\beta_\tau(t) = \gamma_\tau(t) = (\sin t)/\tau,\ \tau = 1, \ldots, \eta, \eta = \max(r,k)$.

We define $A(t, .) : X \to X$ by $(A(t,.)w)(x) = a(x,t,.)w''$ with

(i) The domain $D(A(t,.)) = \{w \in X : w, w'$ are absolutely continuous, $w'' \in X, w(0) = w(\pi) = 0\}$ is dense in the Banach space X and independent of t. Then

$$A(t,u)w = \sum_{n=1}^{\infty} n^2 (w, w_n) w_n, \quad w \in D(A),$$

where $(.,.)$ is the inner product in $L^2[0,\pi]$ and $w_n = Z_n \circ u$ is the orthogonal set of eigenvectors in $A(t,u)$, where $Z_n(t,s) = \sqrt{\frac{2}{\pi}} \sin n(t-s)^\alpha,\ 0 < \alpha \le 1, 0 \le s \le t \le a, n = 1, 2, \ldots$.

(ii) The operator $[A(t,.) + \lambda^\alpha I]^{-1}$ exists in $L(X)$ for any λ with $\operatorname{Re} \lambda \le 0$ and

$$\|[A(t,.) + \lambda^\alpha I]^{-1}\| \le \frac{C_\alpha}{|\lambda| + 1}, \quad t \in J.$$

(iii) There exist constants $\eta \in (0,1]$ and C_α such that

$$\|[A(t_1,.) - A(t_2,.)]A^{-1}(s,.)\| \le C_\alpha |t_1 - t_2|^\eta, \quad t_1, t_2, s \in J.$$

Under these conditions each operator $-A(s,.), s \in J$ generates an evolution operator $\exp(-t^\alpha A(s,.)), t > 0$ (which is compact, analytic and self-adjoint) and there exists a constant C_α such that

$$\|A^n(s,.) \exp(-t^\alpha A(s,.))\| \le \frac{C_\alpha}{t^n},$$

where $n = 0, 1, t > 0, s \in J$, (compare with [38])

In particular, we conclude that, the evolution operator in fact is an (α, u)-resolvent family has the form:

$$R_{(\alpha,u)}(t,s)w = \sum_{n=1}^{\infty} \exp[-n^2 (t-s)^\alpha](w, w_n) w_n, \quad w \in X.$$

Assume that the linear operator W is given by

$$\begin{aligned} W\mu(x) &= \int_0^a R_{(\alpha,u)}(a,s) B\mu(x,s) \mathrm{d}s \\ &= \sum_{n=1}^{\infty} \int_0^a \exp[-n^2(a-s)^\alpha](\zeta(x,s), w_n) w_n \mathrm{d}s, \quad x \in [0,\pi] \end{aligned}$$

has a bounded invertible operator $\tilde{W}^{-1}$ in $L^2(J,U)/\ker W$.

Further all assumptions (H_1)–(H_7) are satisfied and it is possible to choose our constants in H_8. Hence by Theorem 3.3 system (4.1)–(4.4) is controllable on J.

5. Conclusion

In this article, the controllability result for a class of fractional evolution nonlocal impulsive quasilinear multi-delay integro-differential systems in a Banach space has been considered. A new set of sufficient conditions are derived for our main result by using the theory of fractional calculus, fixed point technique and (α, u)-resolvent family (a new concept).

References

[1] D. Baleanu, O.G. Mustafa, On the global existence of solutions to a class of fractional differential equations, Computers & Mathematics with Applications 59 (5) (2010) 1835–1841.
[2] D. Baleanu, A.K. Golmankhaneh, On electromagnetic field in fractional space, Nonlinear Analysis: Real World Applications 11 (1) (2010) 288–292.
[3] D. Baleanu, J.I. Trujillo, A new method of finding the fractional Euler–Lagrange and Hamilton equations within Caputo fractional derivatives, Communications in Nonlinear Science and Numerical Simulation 15 (5) (2010) 1111–1115.
[4] R.P. Agarwal, V. Lakshmikantham, J.J. Nieto, On the concept of solution for fractional differential equations with uncertainty, Nonlinear Analysis 72 (2010) 2859–2862.
[5] V. Lakshmikantham, Theory of fractional functional differential equations, Nonlinear Analysis 69 (2008) 3337–3343.
[6] V. Lakshmikantham, A.S. Vatsala, Basic theory of fractional differential equations, Nonlinear Analysis 69 (2008) 2677–2682.
[7] V. Lakshmikantham, A.S. Vatsala, General uniqueness and monotone iterative technique for fractional differential equations, Applied Mathematics Letters 21 (2008) 828–834.
[8] V. Lakshmikantham, J. Vasundhara Devi, Theory of fractional differential equations in Banach spaces, European Journal of Pure and Applied Mathematics 1 (2008) 38–45.
[9] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and applications of fractional differential Equations, in: North-Holland Mathematics Studies, vol. 204, Elsevier Science B.V, Amsterdam, 2006.
[10] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives: Theory and Applications, Gordon and Breach, Yverdon, 1993.
[11] A. Debbouche, Fractional evolution integro-differential systems with nonlocal conditions, Advances in Dynamical Systems and Applications 5 (1) (2010) 49–60.
[12] A. Debbouche, M.M. El-Borai, Weak almost periodic and optimal mild solutions of fractional evolution equations, Electronic Journal of Differential Equations 46 (2009) 1–8.
[13] M.M. El-Borai, K.E. El-Nadi, E.G. El-Akabawy, On some fractional evolution equations, Computers & Mathematics with Applications 59 (2010) 1352–1355.
[14] N.-e. Tatar, On a boundary controller of fractional type, Nonlinear Analysis 72 (2010) 3209–3215.
[15] G.M. N'Guérékata, A Cauchy problem for some fractional abstract differential equation with non local conditions, Nonlinear Analysis 70 (2009) 1873–1876.
[16] M.A.A. El-Sayeed, Fractional order diffusion wave equation, International Journal of Theoretical Physics 35 (1966) 311–322.
[17] L. Byszewski, Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem, Journal of Mathematical Analysis and Applications 162 (1991) 494–505.
[18] L. Byszewski, Theorems about the existence and uniqueness of continuous solutions of nonlocal problem for nonlinear hyperbolic equation, Applicable Analysis 40 (1991) 173–180.
[19] K. Deng, Exponential decay of solutions of semilinear parabolic equations with nonlocal initial conditions, Journal of Mathematical Analysis and Applications 179 (1993) 630–637.
[20] Z. Tai, X. Wang, Controllability of fractional-order impulsive neutral functional infinite delay integrodifferential systems in Banach spaces, Applied Mathematics Letters 22 (2009) 1760–1765.
[21] V. Lakshmikantham, D.D. Bainov, P.S. Simeonov, Theory of Impulsive Differential Equations, World Scientific, Singapore, 1989.
[22] R. Subalakshmi, K. Balachandran, Approximate controllability of nonlinear stochastic impulsive integrodifferential systems in hilbert spaces, Chaos, Solitons and Fractals 42 (2009) 2035–2046.
[23] K. Balachandran, J.Y. Park, Controllability of fractional integrodifferential systems in Banach spaces, Nonlinear Analysis: Hybrid Systems 3 (2009) 363–367.
[24] K. Balachandran, R. Sakthivel, Controllability of integrodifferential systems in Banach spaces, Applied Mathematics and Computation 118 (2001) 63–71.
[25] R. Sakthivel, Q.H. Choi, S.M. Anthoni, Controllability result for nonlinear evolution integrodifferential systems, Applied Mathematics Letters 17 (2004) 1015–1023.
[26] R. Sakthivel, S.M. Anthoni, J.H. Kim, Existence and controllability result for semilinear evolution integrodifferential Systems, Mathematical and Computer Modelling 41 (2005) 1005–1011.
[27] N. Abada, M. Benchohra, H. Hammouche, Existence and controllability results for nondensely defined impulsive semilinear functional differential inclusions, Journal of Differential Equations 246 (2009) 3834–3863.
[28] M. Benchohra, L. Gorniewicz, S.K. Ntouyas, A. Ouahab, Controllability results for impulsive functional differential inclusions, Reports on Mathematical Physics 54 (2004) 211–228.
[29] R. Sakthivel, E.R. Anandhi, Approximate controllability of impulsive differential equations with state-dependent delay, International Journal of Control 83 (2) (2009) 387–393.
[30] J.M. Jeong, J.R. Kim, H.H. Roh, Controllability for semilinear retarded control systems in Hilbert spaces, Journal of Dynamical and Control Systems 13 (4) (2007) 577–591.
[31] R.P. Agarwal, Y. Zhou, Y. He, Existence of fractional neutral functional differential equations, Computers & Mathematics with Applications 59 (2010) 1095–1100.
[32] R.P. Agarwal, M. Benchohra, S. Hamani, A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions, Acta Applicandae Mathematicae 109 (2010) 973–1033.
[33] M.M. El-Borai, Some probability densities and fundamental solutions of fractional evolution equations, Chaos, Solitons and Fractals 14 (2002) 433–440.
[34] Y. Zhou, F. Jiao, Nonlocal Cauchy problem for fractional evolution equations, Nonlinear Analysis: Real World Applications 11 (2010) 4465–4475.
[35] J.R. Wang, Y. Zhou, A class of fractional evolution equations and optimal controls, Nonlinear Analysis: Real World Applications 12 (2011) 262–272.
[36] D. Araya, C. Lizama, Almost automorphic mild solutions to fractional differential equations, Nonlinear Analysis 69 (2008) 3692–3705.
[37] G.M. Mophou, G.M. N'Guérékata, On some classes of almost automorphic functions and applications to fractional differential equations, Computers and Mathematics with Applications 59 (2010) 1310–1317.
[38] E. Hille, R.S. Phillips, Functional Analysis and Semi-Groups, in: American Mathematical Society Colloquium Publications, vol. 31, American Mathematical Society, Providence, RI, USA, 1957.
[39] I.M. Gelfand, G.E. Shilov, Generalized Functions, Vol. 1, Nauka, Moscow, 1959.
[40] I. Podlubny, Fractional differential equations, Math. in Science and Eng., Vol. 198, Technical University of Kosice, Slovak Republic, 1999.
[41] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, New York, 1983.
[42] S.D. Zaidman, Abstract differential equations, in: Research Notes in Mathematics, Pitman Advanced Publishing Program, San Francisco, London Melbourne, 1979.